Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

```
Clear["Global`*"]
```

2 - 9 Power series
Where does the power series converge uniformly?

3. $\sum_{n=0}^{\infty} \left(\frac{1}{3^n}\right)^n (z + i)^{2n}$

```
Clear["Global`*"]
```

By theorem 1, p. 699, a power series in powers of $z - z_0$ converges uniformly in the closed disk $|z - z_0| \leq r$, where $r < R$ and R is is the radius of convergence of the series. In other words, look for the radius of convergence.

```
Series[(1/3^n)^n (z + i)^(2 n), {n, i, 4}];
```

The power series is in terms of $Z=(z+i)^2$, and has the form $\sum_{n=0}^{\infty} a_n Z^n$ with coefficients $\frac{1}{3^n}$. So

$a_n = 3^{-n}$

3^{-n}

$a_{n+1} = 3^{-(n+1)}$

3^{-1-n}

and

$\frac{a_n}{a_{n+1}}$

3

Since the power of the power term is 2n, the radius of convergence R is

$3^{1/2}$

$\sqrt{3}$

The disk of uniform convergence is less than R, so a δ must be allowed so that $|z+i| \leq \sqrt{3} - \delta$, with $\delta > 0$.

5. $\sum_{n=2}^{\infty}$ Binomial[n, 2] $(4 z + 2 i)^n$

```
Clear["Global`*"]
```

The form of the series can be changed to

```
Sum[Binomial[n, 2] 4^n (z + ⅈ/2)^n, {n, 2, 8}]
```

$$16\left(\frac{i}{2}+z\right)^2+192\left(\frac{i}{2}+z\right)^3+1536\left(\frac{i}{2}+z\right)^4+10240\left(\frac{i}{2}+z\right)^5+61440\left(\frac{i}{2}+z\right)^6+344064\left(\frac{i}{2}+z\right)^7+1835008\left(\frac{i}{2}+z\right)^8$$

And to find the general sequence of coefficients,

```
FindSequenceFunction[{16, 192, 1536, 10240, 61440, 344064, 1835008}, n]
```

$$2^{1+2n}\, n\,(1+n)$$

Reaching back to get the Cauchy-Hadamard criterion, I can find the raw radius,

```
Limit[Abs[2^(1+2n) n (1+n) / (2^(3+2n) (n+1) (2+n))], n → ∞]
```

$$\frac{1}{4}$$

And to convert the raw radius to the actual radius of convergence, I apply the 1/n factor of the power term,

$$\left(\frac{1}{4}\right)^{1/1}$$

$$\frac{1}{4}$$

As the radius of convergence is $R=\frac{1}{4}$, I now need r such that $r+\delta=\frac{1}{4}$, where $\delta>0$ and $\text{Abs}[z+\frac{i}{2}] \le r$.

7. $\sum_{n=1}^{\infty} \frac{n!}{n^2}\left(z+\frac{i}{2}\right)^n$

```
Clear["Global`*"]
```

I see that the Maclaurin series does not converge, but the Taylor series does,

```
Sum[n!/n^2 (z + ⅈ/2)^n, {n, 1, 8}]
```

$$\frac{i}{2}+z+\frac{1}{2}\left(\frac{i}{2}+z\right)^2+\frac{2}{3}\left(\frac{i}{2}+z\right)^3+\frac{3}{2}\left(\frac{i}{2}+z\right)^4+\frac{24}{5}\left(\frac{i}{2}+z\right)^5+20\left(\frac{i}{2}+z\right)^6+\frac{720}{7}\left(\frac{i}{2}+z\right)^7+630\left(\frac{i}{2}+z\right)^8$$

The z and n parts of the series are already set up nicely. I can try to find the coefficients,

```
FindSequenceFunction[{1, 1/2, 2/3, 3/2, 24/5, 20, 720/7, 630}, n]
```

$$\frac{\text{Pochhammer}[1, -1 + n]}{n}$$

```
FullSimplify[%]
```

$$\frac{\text{Gamma}[n]}{n}$$

Using the Cauchy-Hadamard criterion, I can try to find the radius,

```
Limit[Abs[Gamma[n]/n (n + 1)/Gamma[n + 1]], n → ∞]
```

0

The radius of convergence is zero. The disk of uniform convergence must be strictly less than the radius of convergence, which is impossible. Therefore the series is uniformly convergent nowhere.

9. $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n^2} (z - 2 i)^n$

```
Clear["Global`*"]
```

The Maclaurin series does not converge, but the Taylor series does,

```
Sum[(-1)^n/(2^n n^2) (z - 2 i)^n, {n, 1, 10}]
```

$$\frac{1}{2}(2 i - z) + \frac{1}{16}(-2 i + z)^2 - \frac{1}{72}(-2 i + z)^3 + \frac{1}{256}(-2 i + z)^4 - \frac{1}{800}(-2 i + z)^5 + \frac{(-2 i + z)^6}{2304} - \frac{(-2 i + z)^7}{6272} + \frac{(-2 i + z)^8}{16\,384} - \frac{(-2 i + z)^9}{41\,472} + \frac{(-2 i + z)^{10}}{102\,400}$$

The sign needs to be adjusted on the first term in order to match the others. The z and n parts of the series seem neat and orderly. I can try to find the coefficients,

```
FindSequenceFunction[
 {-1/2, 1/16, -1/72, 1/256, -1/800, 1/2304, -1/6272, 1/16384}, n]
```

$$\frac{\left(-\frac{1}{2}\right)^n}{n^2}$$

Using the Cauchy - Hadamard criterion, I can try to find the radius,

```
Limit[Abs[((-1/2)^n/n^2) ((n + 1)^2/(-1/2)^(n+1))], n → ∞]
```

2

And to convert the raw radius to the actual radius of convergence, I apply the 1/n factor of the power term,

$2^{1/1}$

2

As the radius of convergence is R=2, I now need r such that r+δ=2, where δ>0 and Abs[z-2 i] ≤ r.

10 - 17 Uniform convergence
Prove that the series converges uniformly in the indicated region.

11. $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$, Abs[z] ≤ 1

```
Clear["Global`*"]
```

```
SumConvergence[z^n/n^2, n]
```

Abs[z] ≤ 1

Mathematica indicates the region of convergence, which is exactly the region I am interested in. (However, since I want to enforce r and not R, I don't think I can do it on the response by Mathematica.) The Weierstrass M-test is very easy to apply in this case. For the domain of interest, for any z^n with z from that domain,

$\frac{z^n}{n^2} \leq \frac{1}{n^2}$ (* for all positive n∈ℕ *)

And the series $\frac{1}{n^2}$can be used as the Weierstrass comparison series. Example 4 on p. 682 remarks that the series $\frac{1}{n^2}$ converges. Therefore by the Weierstrass M-test, the problem series converges uniformly in the indicated region.

13. $\sum_{n=1}^{\infty} \frac{\text{Sin}[\text{Abs}[z]]^n}{n^2}$, all z

```
Clear["Global`*"]
```

This problem is very similar to the last. The numerator of the function must either equal 1 or be less than 1, for all z. In either case

$$\frac{\text{Sin[Abs[z]]}^n}{n^2} \le \frac{1}{n^2} \text{ (* for all positive } n\in\mathbb{N} \text{ *)}$$

And again the series $\frac{1}{n^2}$ can be used as the Weierstrass comparison series. Example 4 on p. 682 remarks that the series $\frac{1}{n^2}$ converges. Therefore by the Weierstrass M - test, the series converges uniformly in the indicated region.

15. $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} z^n$, $\text{Abs}[z] \le 3$

```
Clear["Global`*"]
```

$\text{SumConvergence}\left[\frac{(n!)^2}{(2n)!} z^n, n\right]$

$\text{Abs}[z] < 4$

Mathematica tells me that the radius of convergence, R, is equal to 4. Therefore, since the problem series is a power series, the radius of uniform convergence, r, is anything smaller than R. As 3 is certainly less than 4, the series converges uniformly in the indicated region.

17. $\sum_{n=1}^{\infty} \frac{\pi^n}{n^4} z^{2n}$, $\text{Abs}[z] \le 0.56$

```
Clear["Global`*"]
```

$\text{ver} = \frac{\pi^n}{n^4} z^{2n}$

$\frac{\pi^n z^{2n}}{n^4}$

```
ver1 = ver /. z → 0.56; ver2 = ver /. z → 0.1; ver3 = ver /. z → 0.0;
ver4 = ver /. z → -0.1; ver5 = ver /. z → -0.56;
```

```
TableForm[Table[{n, NumberForm[ver1, 3], NumberForm[ver2, 3],
   ver3, NumberForm[ver4, 3], NumberForm[ver5, 3],
   NumberForm[N[1/n^2], 3]}, {n, 1, 10}], TableHeadings → {{},
   {"", "z→0.56", "z→0.1", "z→0.0", "z→-0.1", "z→-0.56", "N[1/n^2]"}}]
```

	z→0.56	z→0.1	z→0.0	z→-0.1	z→-0.56	N[
1	0.985	0.0314	0.	0.0314	0.985	1.
2	0.0607	0.0000617	0.	0.0000617	0.0607	0.[illegible]
3	0.0118	3.83×10^{-7}	0.	3.83×10^{-7}	0.0118	0.[illegible]
4	0.00368	3.81×10^{-9}	0.	3.81×10^{-9}	0.00368	0.[illegible]
5	0.00149	4.9×10^{-11}	0.	4.9×10^{-11}	0.00149	0.[illegible]
6	0.000706	7.42×10^{-13}	0.	7.42×10^{-13}	0.000706	0.[illegible]
7	0.000375	1.26×10^{-14}	0.	1.26×10^{-14}	0.000375	0.[illegible]
8	0.000217	2.32×10^{-16}	0.	2.32×10^{-16}	0.000217	0.[illegible]
9	0.000133	4.54×10^{-18}	0.	4.54×10^{-18}	0.000133	0.[illegible]
10	0.0000862	9.36×10^{-20}	0.	9.36×10^{-20}	0.0000862	0.[illegible]

The familiar series $\frac{1}{n^2}$ can be used as the Weierstrass comparison series. Example 4 on p. 682 remarks that the series $\frac{1}{n^2}$ converges. Using this series, the Weierstrass M - test demonstrates convincingly that the series

$$\frac{\pi^n}{n^4} z^{2n} < \frac{1}{n^2}$$

by use of a sequence of successive values, with difference gap opening on increasing n, and that it therefore converges uniformly in the indicated region. (Note: I wanted to use **Solve** or **Reduce** to make a better case, but neither was able to come through with something useful.)